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On the connection between hyperelliptic separability and Painlevé integrability

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Abstract

We consider systems of ODEs which are associated with some physically significant examples: shallow water equilibrium solutions, travelling waves of the Harry Dym equation, a Lotka–Volterra system of competing species and the geodesic flow on the triaxial ellipsoid. The first three are shown to share the following properties: (i) they are hyperelliptically separable systems (HSS) and, after a suitable nonlinear time transformation, become algebraically completely integrable (ACI) and (ii) they are of the weak Painlevé type and become full Painlevé after the application of this transformation. The geodesic flow on the other hand, although it passes the usual Painlevé test, does not possess a full set of free constants and thus one may not conclude whether it has the Painlevé property or not. This system is also HSS and becomes ACI after the application of a suitable nonlinear time transformation. We also combine our geometric-analytical investigation with a numerical analysis of the system in the complex plane and show that there is perfect correspondence between the results of the two approaches. This correspondence strengthens the reliability of such numerical studies and helps us better understand their implication in cases where such nonlinear transformations to complete integrability are not available.

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1. Introduction

One of the main questions concerning nonlinear dynamical systems is the implementation of effective criteria to decide their integrability or non-integrability from the knowledge of the singularity structure of the solutions [1]. As is well known, in many physical interesting cases this is indeed possible and has led many researchers to associate the complete integrability of a system with the Painlevé property, i.e. that the solutions are free from movable critical points [2]. However, in other integrable systems this so-called strong Painlevé property does not hold and a weaker property has been introduced.

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We recall that the strong (or full) Painlevé property means that the system admits near every singularity formal Laurent series expansions of which at least one family depends on the maximal number of free parameters [3]. On the other hand, for the weak Painlevé property, Puisseux series need to be developed as, near some of its singularities, the solutions are allowed to contain rational powers of the independent variables [4] (see also [5] for a global definition of the weak Painlevé property).

The relation between the full Painlevé property and algebraic complete integrability (ACI) has long been investigated starting with the papers of Adler and van Moerbeke [6]. More recently, the connection between the weak Painlevé property and integrability has also been explained in the case of hyperelliptically separable systems (HSS) [7].

In this paper we study four physically significant examples: time independent solutions of a shallow water partial differential equation (PDE), wave solutions of the Harry Dym PDE, an integrable Lotka–Volterra system and the geodesic flow on a triaxial ellipsoid. We prove that the first three are naturally HSS and, after a suitable nonlinear time transformation (NTT), become ACI. Furthermore, we show that all of them, apart from the geodesic flow, are weak Painlevé and become full Painlevé after the application of this transformation.

In the case of the geodesic flow, we find that, although it passes the standard Painlevé test, it does not possess a full set of free constants, and thus is not ACI. Nevertheless, we show that it is HSS as it separates in ellipsoidal coordinates and becomes ACI after the application of a suitable NTT.

In particular, we demonstrate in the above examples that, in the starting variables, the generic invariant manifolds are open subsets of g-dimensional strata of a convenient $(g+1)$-dimensional generalized Jacobian associated to a hyperelliptic curve Γ of genus g. After the NTT the system is algebraically completely integrable in the new variables and the generic invariant manifold is an open subset of the Jacobian of Γ itself.

We also combine in this paper our geometric-analytical investigation with a numerical study of the solutions in the complex plane and demonstrate that there is a perfect correspondence between the two approaches. This is important because it strengthens the reliability of such numerical studies and leads to a correct interpretation of their results, even in cases where a similar geometric-analytical study is not available.

To make our paper self-contained we provide in section 2 a brief outline of the main concepts and ideas of hyperelliptic separability and ACI.

Other authors (e.g. Goriely [8] in particular), have already observed that under certain NTTs a system can pass from weak to full Painlevé and we find here that Goriely's algorithm works for our examples. Of course, our investigation leads to the natural question whether it is possible to identify the precise conditions under which a system becomes full Painlevé after applying a NTT. We conjecture that such transformations are successful, i.e. they allow one to pass from a weak to a full Painlevé property, in all cases where a geometric picture similar to our examples holds.

This question is not only of theoretical interest, since, in general, the solutions of systems which have the weak Painlevé property are infinitely sheeted and 'badly' defined as functions, as we explain in section 3 in the case of a Henon–Heiles quartic potential. On the other hand, solutions of ACI systems are meromorphic functions. So it is clear that, if a NTT is available, one can get, at least locally, some more detailed information about the original system by writing down 'good' solutions in the new time variable and then applying them to it.

In section 4 we give the connection between the NTT and hyperelliptic separability, while sections 5–8 describe our results for the shallow water solutions, the Harry Dym travelling waves, the Lotka–Volterra system and the geodesic flow, respectively. Finally, we discuss our conclusions in section 9.

2. Algebraic complete integrability and hyperelliptically separable systems

The examples we consider in this paper all satisfy the following geometric description: assume that we start with an integrable system of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \qquad x = (x_1, \ldots, x_n)$$

with first integrals $I_1, \ldots, I_r$ and after a convenient change of variables $x = T(\nu)$ and fixing the constants of motion, $I_j = c_j$, $j = 1, \ldots, r$, we obtain equations of the following type:

$$\frac{\nu_1^{l(k)}\,\mathrm{d}\nu_1}{\sqrt{R(\nu_1)}} + \cdots + \frac{\nu_N^{l(k)}\,\mathrm{d}\nu_N}{\sqrt{R(\nu_N)}} = \mathrm{d}\phi_k \qquad k = 1, \ldots, N \tag{1}$$

where $\mathrm{d}\phi_k = \mathrm{d}_k\,\mathrm{d}t$ (d_k constants),

$$R(\nu) = A\prod_{j=1}^{s}(\nu - e_j) \qquad A = A(c_1, \ldots, c_r) \quad e_j = e_j(c_1, \ldots, c_r)$$

and $l(k)$ is a non-negative integer power, such that $l(k-1) < l(k)$.

The above picture naturally leads to the introduction of the genus g hyperelliptic curve Γ, whose affine part satisfies the equation

$$\Gamma = \{(\nu, w) \;:\; w^2 = R(\nu)\}$$

where the genus is defined by $s = 2g+1,\ 2g+2$. Let us fix a canonical basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ and a vector $\omega = (\omega_1, \ldots, \omega_g)^T$ of independent holomorphic differentials on Γ. In particular, the following basis of holomorphic Abelian differentials

$$\omega_j = \frac{\nu^{j-1}\mathrm{d}\nu}{R(\nu)} \qquad j = 1, \ldots, g$$

plays an important role in the examples we consider in the next sections. We recall that $\mathrm{Jac}(\Gamma) = \boldsymbol{C}^g/\Lambda$, where $C^g = (\phi_1, \ldots, \phi_g)$ and Λ is the lattice generated by the $2g$ period vectors of ω along the canonical cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$.

Fix P_0 on Γ (basepoint). If we denote by $\Gamma^{(n)}$ the n symmetric power of Γ, the complete image of the map

$$\mathcal{A} : \Gamma^{(n)} \to \mathrm{Jac}(\Gamma) \qquad \mathcal{A}(P_1, \ldots, P_n) = \sum_{i=1}^{n}\int_{P_0}^{P_i}\omega$$

is an n-dimensional stratum of the Jacobian of Γ, $W_n = \mathcal{A}(\Gamma^{(n)})$ and, in particular, $W_g \equiv \mathrm{Jac}(\Gamma)$.

If $N = g$ and the differentials in (1) form a basis of holomorphic differentials, then (1) are Abel–Jacobi equations in differential form, and the complete image of $\mathcal{A}(\Gamma^{(g)})$ is $\mathrm{Jac}(\Gamma)$, $\phi_1, \ldots, \phi_g$ depend linearly on t and, by the Jacobi inversion problem, $x(\phi_1, \ldots, \phi_g)$ are meromorphic. We call such a system ACI.

If $N < g$ and all the differentials in (1) are holomorphic, the complex invariant manifolds are N-dimensional strata of the g-dimensional $\mathrm{Jac}(\Gamma)$ and only $\phi_1, \ldots, \phi_N$ evolve linearly in time, while the remaining 'angles' $\phi_{N+1}, \ldots, \phi_g$ depend analytically and nonlinearly on $\phi_1, \ldots, \phi_N$ as well as on t. We call such a system HSS. There is little hope in this case that the solution $x(t)$ may be well defined globally and indeed, as a consequence of the Jacobi inversion theorem, we may conclude in general that it is an infinitely sheeted map [7]. In the next section, we consider a Henon–Heiles Hamiltonian which is indeed HSS and present numerical evidence of its infinitely sheeted solutions.

In finite-dimensional reductions of the water wave equation and the Harry Dym equation, and in the geodesic flow on the ellipsoid (treated in sections 5, 6 and 8, respectively) the following differential:

$$\frac{\nu^g \mathrm{d}\nu}{R(\nu)}$$

also appears in (1). It is an Abelian differential of the second kind with a double pole at infinity, if $s = 2g + 1$, or an Abelian differential of the third kind with simple poles at $\pm\infty$, if $s = 2g + 2$. In such cases, we can generalize the above picture in the following way: let Ω^j_P denote the canonical differential of the second kind with a pole of order j at $P \in \Gamma$ and Ω_{PQ} the differential of the third kind with simple poles at P and Q and residues, respectively, ± 1. Then a $(g + 1)$-dimensional generalized Jacobian is $\mathrm{Jac}^*(\Gamma) = C^{g+1}/\Lambda^*$, where Λ^* is the lattice generated by the $2g$ period vectors of $\omega^* = (\omega, \Omega^j_P)^T$ along the canonical cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ (or, the lattice generated by the $2g+1$ period vectors of $\omega^* = (\omega, \Omega_{PQ})^T$ along the canonical cycles $a_1, \ldots, a_g, b_1, \ldots, b_g, \gamma_P$, where γ_P denotes a closed cycle around P which does not include Q and does not intersect $a_1, \ldots, a_g, b_1, \ldots, b_g$).

As before, we may define strata on the generalized Jacobian as follows: let P_0 and $\Gamma^{(n)}$ be defined as earlier, (respectively $\Gamma^{(n)}_* = \Gamma^{(n)} \setminus \{P_i = P, Q \text{ for some } i = 1, \ldots, n\}$). Then the complete image of the following generalization of the Abel–Jacobi map:

$$\mathcal{A}^* : \Gamma^{(n)} \to \mathrm{Jac}^*(\Gamma) \qquad \text{resp.} \qquad \mathcal{A}^* : \Gamma^{(n)}_* \to \mathrm{Jac}^*(\Gamma)$$

defined as

$$\mathcal{A}^*(P_1, \ldots, P_n) = \sum_{i=1}^{n} \int_{P_0}^{P_i} \omega^*$$

is an n-dimensional stratum of the generalized Jacobian of Γ, $W^*_n = \mathcal{A}^*(\Gamma^{(n)})$ (respectively, $W^*_n = \mathcal{A}^*(\Gamma^{(n)}_*)$) and $W^*_{g+1} = \mathrm{Jac}^*(\Gamma)$.

If $N = g + 1$ in (1) and if the differentials of the first g equations in (1) form a basis of holomorphic differentials, while a meromorphic differential appears in the last equation, we conclude that the generic invariant manifold of the system is an open subset of $\mathrm{Jac}^*(\Gamma)$ and we call such a system ACI.

If $N < g + 1$, in all other cases, the system is HSS and the generic invariant manifold is an open subset of an N-dimensional stratum W^*_N of $\mathrm{Jac}^*(\Gamma)$. In sections 5 and 6 we show that the finite-dimensional reductions of the shallow water wave and the Harry Dym equations fall into this class.

3. A Henon–Heiles system

In this section, we shall use as an illustration of our approach the Henon–Heiles Hamiltonian system:

$$\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= 2xy^3 + \frac{3}{4}x^3 y \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= 5y^4 + 3x^2y^2 + \frac{3}{16}x^4. \end{aligned} \tag{2}$$

This is an example where an additional independent algebraic integral exists and is in involution with the Hamiltonian, thus the system is integrable in the sense of Arnol'd–Liouville [1, 4]. However, it is also a case where the system admits algebraic singularities and, as we show later, it is indeed hyperelliptically separable. Moreover, Goriely's approach [8] of a NTT, as described in section 4, is unable to transform this system into a full Painlevé one.

Observe that system (2) is separable in parabolic coordinates

$$x^2 = -4\nu_1\nu_2 \qquad y = \nu_1 + \nu_2$$

in which the Hamiltonian takes the Stäckel form

$$H^{(1)}(p_1, p_2, \nu_1, \nu_2) = \frac{\nu_1}{2(\nu_1 - \nu_2)} {p_1}^2 - \frac{\nu_2}{2(\nu_1 - \nu_2)} {p_2}^2 - \frac{{\nu_1}^6 - {\nu_2}^6}{\nu_1 - \nu_2}$$

where p_i denotes here the conjugate momentum to ν_i, $i = 1, 2$. In fact, this system possesses a second integral independent of (and in involution with) $H^{(1)}$:

$$H^{(2)}(p_1, p_2, \nu_1, \nu_2) = \frac{\nu_1\nu_2}{2(\nu_1 - \nu_2)} ({p_2}^2 - {p_1}^2) + \frac{\nu_1\nu_2({\nu_1}^5 - {\nu_2}^5)}{\nu_1 - \nu_2}.$$

By fixing the constants of motion $H^{(i)} = c_i$, $i = 1, 2$ and using the relation

$$\dot{\nu}_i \equiv \frac{d\nu_i}{dt} = \frac{\partial H^{(1)}}{\partial p_i} = (-1)^{i-1} \frac{\nu_i}{\nu_1 - \nu_2} p_i \qquad i = 1, 2$$

system (2) can be reduced to the equations

$$\begin{aligned} &\frac{d\nu_1}{\sqrt{2R(\nu_1)}} + \frac{d\nu_2}{\sqrt{2R(\nu_2)}} = 0 \\ &\frac{\nu_1\, d\nu_1}{\sqrt{2R(\nu_1)}} + \frac{\nu_2\, d\nu_2}{\sqrt{2R(\nu_2)}} = dt \\ &R(\nu) = \nu(c_2 + c_1\nu + \nu^6) \end{aligned} \tag{3}$$

where the differentials in (3) are holomorphic differentials associated to the genus three hyperelliptic curve:

$$\Gamma : \{(\nu, w) \ : \ w^2 = R(\nu)\}.$$

Thus, by comparison with the definitions of the previous section, we may immediately conclude that this is an example of an HSS since the generic invariant manifolds are open subsets of two-dimensional strata $W_2 \subset \mathrm{Jac}(\Gamma)$.

We now demonstrate that numerical investigation in the complex time plane also indicates that this system has ISS (infinitely sheeted solutions). To do this we will integrate equations (2) in the complex t-plane using the following procedure: taking as initial conditions $x(0) = 1$, $\dot{x}(0) = 0.5$, $y(0) = 1$ and $\dot{y}(0) = 0$ (dot denotes differentiation with respect to t), we integrate clockwise along the rectangular contour $-0.6 \leqslant \mathrm{Re}\,(t) \leqslant 0.6$ and $-1.7 \leqslant \mathrm{Im}\,(t) \leqslant 1.7$ (this procedure will be further explained in section 5). We find that four singularities enter the contour and x returns to its initial value after one turn.

We now keep everything the same, but change the integration path: first, we calculate $x_P(0)$ (the initial value of x at $P = (0, 0.01)$). Then we integrate clockwise along the up-right rectangular contour, as shown in figure 1. We then integrate clockwise along the up-left, down-left and down-right contours and after every such four turns we calculate the difference $\Delta x_P(N) = |x_P(N) - x_P(0)|$, $N = 1, 2, \ldots$. In this way we discover a dense pattern of singularities (figure 2(*a*)) and x does not return to its initial value (figure 2(*b*)), even after $N = 100$ (or more) turns.

Observe that if one expands the solutions of (2) near a movable singularity $t = t_*$ in the complex t-plane, one obtains asymptotic expansions of $x(t)$, $y(t)$ in powers of $(t - t_*)^{1/3}$, indicating that this system has only algebraic singularities and is of the weak Painlevé type [1,4].

Let us also remark that had we not changed to a more complicated path (as described above), but kept integrating exclusively along the clockwise direction, we might have missed the ISS, thus concluding that the system possesses only finitely sheeted solutions (FSS). In fact, an implication of such a wrong result was made in an earlier publication by one of the authors [9].

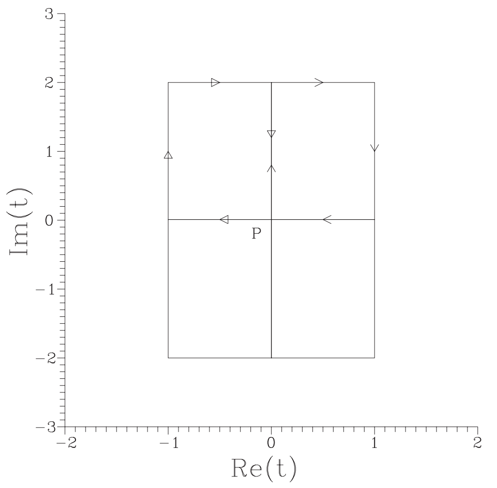

Figure 1. The integration path for the Henon–Heiles Hamiltonian system (2).

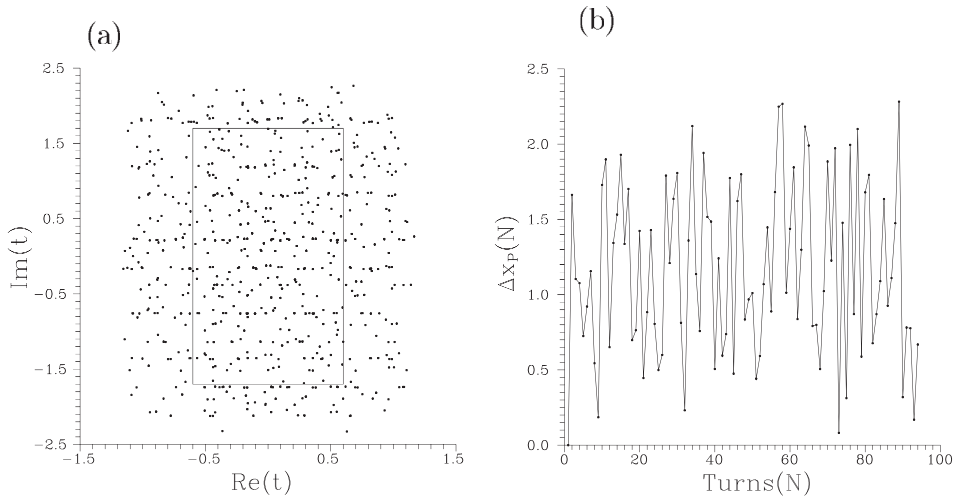

Figure 2. Evidence of ISS for (2) following the integration path shown in figure 1.

4. A nonlinear time transformation

In [8] Goriely introduced a NTT which may transform a weak Painlevé system into one with the full Painlevé property. In particular, let us suppose that we have a system of ODEs of the following form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = x_i \sum_{j=1}^{m} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad i = 1, \ldots, n \tag{4}$$

with $A_{ij}, B_{jk} \in \mathbb{R}$. Then the relation

$$\mathrm{d}t = \mathrm{d}\tilde{t} \prod_{i=1}^{n} x_i^{\beta_i}$$

transforms (4) into the system

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tilde{t}} = x_i \sum_{j=1}^{m} \tilde{A}_{ij} \prod_{k=1}^{n} x_k^{\tilde{B}_{jk}} \qquad i = 1, \ldots, n \tag{5}$$

where

$$\tilde{A} = A \qquad \tilde{B}_{ij} = B_{ij} + \beta_j.$$

If p_i and q_i are, respectively, the powers of the dominant and non-dominant behaviour of (4), r are the resonances (except from -1) and

$$c = -\sum_{i=1}^{n} p_i \beta_i$$

then this NTT gives

$$\tilde{p}_i = \frac{p_i}{1+c} \qquad \tilde{q}_i = \frac{q_i - c}{1+c} \qquad \tilde{r} = \frac{r}{1+c}$$

for system (5). Obviously if system (4) is weak Painlevé with natural denominator d a suitable choice of c is $1 + c = 1/d$ [8].

In the next sections, we show that such a transformation is indeed successful for certain finite-dimensional reductions of the shallow water wave and Harry Dym equations. More generally the NTT is successful in all cases of HSS whose complex invariant manifolds can be completed to N-dimensional strata of $(N+1)$-dimensional generalized Jacobians associated to the genus N hyperelliptic curve

$$\Gamma : \{(\nu, w) \ : \ w^2 = R(\nu)\}$$

and such that

$$\sum_{i=1}^{N} \frac{{\nu_i}^k \, \mathrm{d}\nu_i}{\sqrt{R(\nu_i)}} = \begin{cases} 0 & \text{if} \quad k = 1, \ldots, N-1 \\ \mathrm{d}t & \text{if} \quad k = N. \end{cases}$$

Indeed if we insert the following time transformation:

$$\mathrm{d}T = \mathrm{d}t \prod_{i=1}^{N} \nu_i$$

in the differential equations (4) we get

$$\sum_{i=1}^{N} \frac{{\nu_i}^k \, \mathrm{d}\nu_i}{\sqrt{R(\nu_i)}} = \begin{cases} \mathrm{d}T & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k = 1, \ldots, N-1 \end{cases}$$

which are the differential Abel–Jacobi equations associated to the same hyperelliptic curve. Notice that the NTT leaves the curve invariant and transforms the meromorphic differential to the 'missing' holomorphic differential associated to Γ. As a result, after this transformation, the system turns out to be ACI as expected.

5. The shallow water wave equation

Recently, a new PDE has been proposed for shallow water waves:

$$u_t + u_x - \tfrac{3}{2}\rho_2 \beta u_{xxt} + \left(1 - \tfrac{3}{2}\rho_2\right)\beta u_{xxx} + \alpha u u_x - \tfrac{1}{2}\rho_2 \alpha\beta (u u_{xxx} + 2u_x u_{xx}) = 0 \tag{6}$$

as an improvement of the KdV equation, based on physical grounds [10, 11]. Equation (6) was first derived in [12] by using the method of bi-Hamiltonian systems. It was also studied in [13]

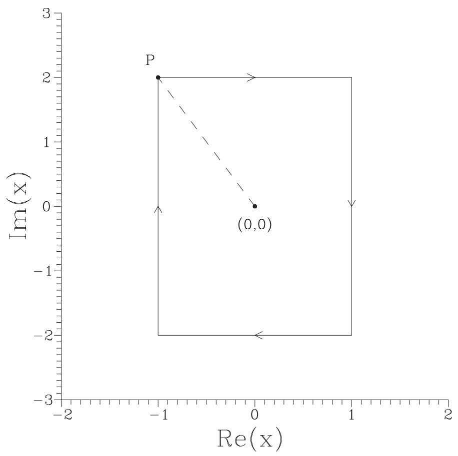

Figure 3. Integration path for equation (7).

from the point of view of Painlevé analysis. Considering equilibrium (i.e. time-independent) solutions of this equation by setting $u = u(x)$ and integrating (6) with respect to x we obtain

$$u + \left(1 - \tfrac{3}{2}\rho_2\right)\beta u_{xx} + \tfrac{1}{2}\alpha u^2 - \tfrac{1}{4}\alpha\beta\rho_2({u_x}^2 + 2uu_{xx}) = c \tag{7}$$

where c is an integration constant (we also assume that $\alpha, \beta, \rho_2 \neq 0$).

Equation (7) has been also studied in [14], from the point of view of singularity analysis in the complex x-plane. We briefly recall here the main results: first, the application of the usual Painlevé test to (7) yields as leading order behaviour of the solutions $u = a_1 + a_2\chi^p$, where $\chi = x - x_*$ (x_* is the singularity), $p = 2/3$, $a_1 = (2-3\rho_2)/(\alpha\rho_2)$ and a_2 is an arbitrary parameter. Free constants enter at resonances -1, 0.

Let us now perform a numerical integration of (7) using the ATOMFT package to study its analytic properties [15–17]. Setting $c = 0$, $\alpha = \rho_2 = 2$ and $\beta = 1$ and taking as initial conditions $u(0) = 1$ and $u_x(0) = 0.1$, we calculate $u_P(0)$ (the initial value of u at point P), using analytic continuation, along a path shown in figure 3. We then integrate equation (7) numerically in the complex x-plane along rectangular contours, starting from P and going clockwise. After each turn we calculate the difference $\Delta u_P(N) = |u_P(N) - u_P(0)|$, where N is the number of turns, as explained in section 3.

Keeping $-1 \leqslant \mathrm{Re}\,(x) \leqslant 1$ and increasing the limits of $\mathrm{Im}\,(x)$, we observe that new singularities enter the contour, but always yield exact returns to the initial conditions after a number of turns N, which increases as the limits of $\mathrm{Im}\,(x)$ are increased (see figures 4(*a*), (*b*), where $-3.4 \leqslant \mathrm{Im}\,(x) \leqslant 3.4$). Similar results are found for different contours. This behaviour, referred to as FSS, constitutes a numerical indication of integrability if it persists for arbitrarily large contours and arbitrary integration paths. Thus, to minimize the possibility of misinterpretation we integrated (7) along a variety of paths of different sizes and always recovered upon returns to the starting point the same initial conditions to very high accuracy. This suggests that (7) may indeed turn into ACI under a suitable nonlinear transformation.

Motivated by this evidence, we were able to prove that the implications of our numerical study are indeed true [14]. To see this, set

$$w(x) = 2\delta u(x) + e \qquad \delta = \tfrac{1}{4}\alpha\beta\rho_2 \quad e = \left(\tfrac{3}{2}\rho_2 - 1\right)\beta$$

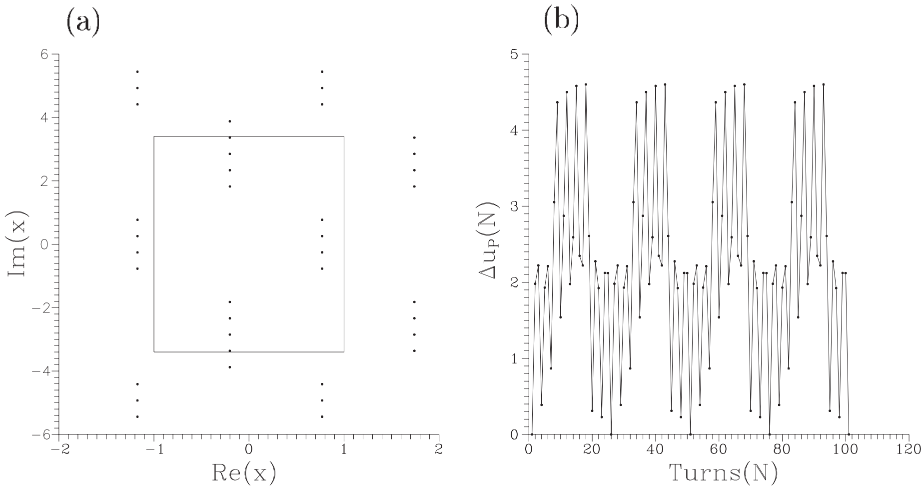

Figure 4. Evidence of FSS for equation (7).

whence equation (7) becomes

$$w\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2-\frac{\alpha}{4\delta}w^2+\left(\frac{\alpha e}{2\delta}-1\right)w+K=0 \tag{8}$$

with

$$K=2\delta c+e-\frac{\alpha e^2}{4\delta}.$$

Then we introduce

$$w(x)=W(t) \qquad \mathrm{d}x=W(t)\,\mathrm{d}t \qquad Z=t\sqrt{\frac{\alpha}{6\delta}}$$

and obtain from (8) the equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}Z^2}=\frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2+\frac{3W^3}{2}+\left(\frac{6\delta}{\alpha}-3e\right)W^2+\left(\frac{3e^2}{2}-\frac{6\delta e}{\alpha}-\frac{12\delta^2c}{\alpha}\right)W$$

which is Painlevé XXX [18].

Let us now observe that equation (8) leads to an HSS if it is written in Hamiltonian form with coordinate w and conjugate momentum $p=w\frac{\mathrm{d}w}{\mathrm{d}x}$:

$$H=\frac{p^2}{2w}+Kw+\frac{1}{2}\left(\frac{\alpha e}{2\delta}-1\right)w^2-\frac{\alpha}{12\delta}w^3.$$

Fixing the constant of motion $H=E$, we immediately obtain

$$\mathrm{d}x=\frac{w\,\mathrm{d}w}{\sqrt{2w(E-Kw+\frac{1}{2}(1-\frac{\alpha e}{2\delta})w^2+\frac{\alpha}{12\delta}w^3)}}. \tag{9}$$

Let Γ be the elliptic curve of the equation

$$\gamma^2=2w\left(E-Kw+\frac{1}{2}\left(1-\frac{\alpha e}{2\delta}\right)w^2+\frac{\alpha}{12\delta}w^3\right).$$

Then relation (9) is an Abelian differential of the third kind, with a pole at $\pm\infty$. The compactified complex generic invariant manifold associated with this system is a one-dimensional stratum of the two-dimensional generalized Jacobian $\mathrm{Jac}^*(\Gamma)=\mathrm{Jac}(\Gamma)\times\boldsymbol{C}^*$. Thus, the reduced system is hyperelliptically separable by definition.

Finally, using the NTT

$$w(x) = W(t) \qquad \mathrm{d}x = W(t)\,\mathrm{d}t$$

relation (9) becomes, in the new time variable t, a differential of the first kind for Γ:

$$\mathrm{d}t = \frac{\mathrm{d}W}{\sqrt{2W(E - KW + \frac{1}{2}(1 - \frac{\alpha e}{2\delta})W^2 + \frac{\alpha}{12\delta}W^3)}}.$$

Thus, in this new time variable, $W(t)$ is meromorphic in t, the associated complex invariant manifold is an open subset of the Jacobian Jac(Γ) and the system is algebraic completely integrable.

Let us observe, however, that this NTT does not preserve the Hamiltonian structure of the system: indeed, the Hamiltonian in W, Π, with Π conjugate momentum, becomes

$$H^1 = \frac{1}{2}\Pi^2 W + KW + \frac{1}{2}\left(\frac{\alpha e}{2\delta} - 1\right)W^2 - \frac{\alpha}{12\delta}W^3$$

and the relation between the old and new conjugate variables is

$$\mathrm{d}p \wedge \mathrm{d}w = W\,\mathrm{d}\Pi \wedge \mathrm{d}W.$$

6. The Harry Dym equation

The Harry Dym equation, as studied in [19], has the form

$$H_t = H^3 H_{xxx}.$$

Considering travelling wave solutions of this equation we set $H = H(z) = H(x - t)$ and obtain

$$H^3 \frac{\mathrm{d}^3 H}{\mathrm{d}z^3} = -\frac{\mathrm{d}H}{\mathrm{d}z}. \tag{10}$$

We have taken the speed of these waves $c = 1$, without loss of generality, as this can be easily scaled out of (10).

A direct application of the Painlevé test to equation (10) yields the following results (see also [19]):

Step 1. The leading order behaviour is $H = a_0\zeta^p$ where $\zeta = z - z_*$, $p = 2/3$ and ${a_0}^3 = -9/4$.

Step 2. The resonances are -1, $2/3$ and $4/3$.

Step 3. Substituting the series

$$H = \zeta^{2/3}(a_0 + a_1\zeta^{1/3} + a_2\zeta^{2/3} + a_3\zeta + a_4\zeta^{4/3} + \cdots)$$

we find that $a_1 = a_3 = 0$ and a_2, a_4 are arbitrary.

Thus, we find again only algebraic singularities in the complex ζ-plane and a deeper analysis of integrability is required.

Moreover equation (10) has FSS as can be easily seen numerically by integrating in the complex z-plane. The singularities appear in straight lines and we always have exact returns after any number of turns, see figures 5(*a*), (*b*) for the initial conditions $H(0) = 1$, $H_z(0) = 0.1$, $H_{zz}(0) = 0$ and $-2 \leqslant \mathrm{Re}\,(z) \leqslant 4.5$, $-16 \leqslant \mathrm{Im}\,(z) \leqslant 16$. Note that the pattern

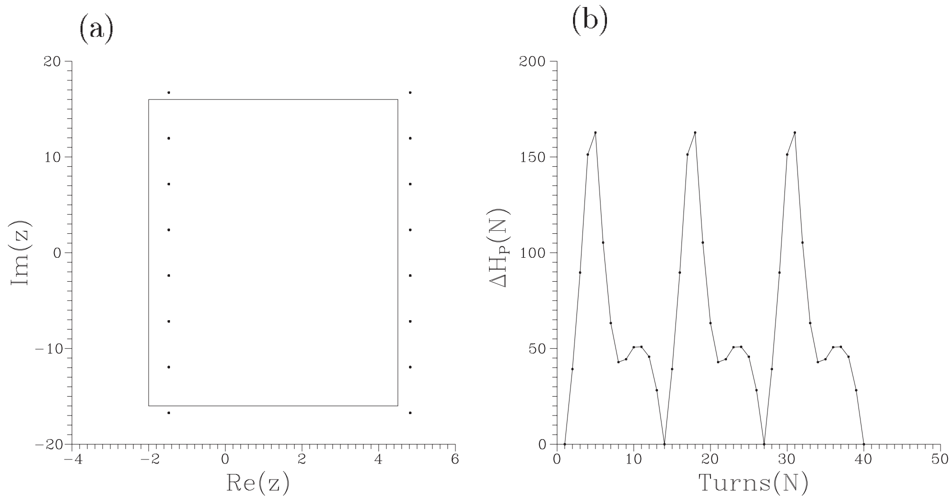

Figure 5. Evidence of FSS for equation (10).

is again periodic as in the case of the shallow water wave equation and any variation of the path does not affect the finitely sheeted structure of the solution.

Let us now show that (10) is indeed integrable, both in Painlevé and algebraic sense. First we set

$$H(z) = w(t) \qquad dz = w(t)\,dt$$

multiply the resulting equation with w^{-1} and integrate once with respect to t to obtain

$$\frac{d^2w}{dt^2} = \frac{3}{2w}\left(\frac{dw}{dt}\right)^2 + cw + 1$$

where c is the integration constant. We then change variables to

$$w(t) = \frac{1}{2cW(Z)} \qquad Z = bt \qquad b^2 = -\frac{c}{2}$$

and the previous equation becomes

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + 4W^2 + 2W$$

which is Painlevé XIX [18].

We also observe that equation (10) may be immediately integrated once to give

$$H\frac{d^2H}{dz^2} - \frac{1}{2}\left(\frac{dH}{dz}\right)^2 - \frac{1}{H} = C$$

with C constant. This last equation can be put in the Hamiltonian form

$$\mathcal{H} = \frac{1}{2}Hp^2 + \frac{C}{H} + \frac{1}{2H^2}$$

where $p = \frac{1}{H}\frac{dH}{dz}$, from which, setting $\mathcal{H} = E$, we immediately get

$$dz = \frac{H\,dH}{\sqrt{2EH^3 - 2CH^2 - H}}$$

which is an Abelian differential of the second kind with a double pole at infinity associated with the elliptic curve:

$$\Gamma : \{\gamma^2 = 2EH^3 - 2CH^2 - H\}.$$

Thus, the complex invariant manifold of this system is a one-dimensional stratum of the two-dimensional generalized Jacobian, $\mathrm{Jac}^*(\Gamma)$. With the time transformation

$$H(z) = w(t) \qquad \mathrm{d}z = w(t)\,\mathrm{d}t$$

we get

$$\mathrm{d}t = \frac{\mathrm{d}w}{\sqrt{2Ew^3 - 2Cw^2 - w}}$$

which is an elliptic differential of the first kind. This proves that $w(t)$ is meromorphic in t in complete agreement with the above reduction to Painlevé XIX and the numerical results. As in the shallow water wave equation the time transformation does not preserve the symplectic structure.

7. The Lotka–Volterra system

A generalized Lotka–Volterra system of three interacting species with equal growth rates for all species, can be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Cxy + xz$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ayz + yx$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = Bxz + zy$$

where we have eliminated the linear growth terms λx, λy, λz from the x, y, z equations, respectively, and A, B, C are constant parameters.

In the case $A = 1$, $BC = 1$, this system is known to admit two integrals [20]

$$\frac{(x - Cy)^2 z^C}{xy} = D^2 \qquad D(x - Cy) - C\int \frac{D^2\,\mathrm{d}z}{\sqrt{D^2 + 4Cz^C}} = E$$

and its equations of motion become

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Cxy + xz$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = yz + yx \tag{11}$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{C}xz + zy.$$

The application of the Painlevé test to (11) yields the following results:

Step 1. The system has the following branches of singular behaviour:

(i) $x = a_0\tau^{-1}$, $y = b_0\tau^{-1}$ and $z = c_0\tau^{-1}$, where $\tau = t - t_*$ and $a_0 = -C/2$, $b_0 = -1/2$, $c_0 = (C - 2)/2$.

(ii) $x = a_0\tau^p$, $y = b_0\tau^{-1}$ and $z = c_0\tau^{-1}$, where $p = -1 - C > -1$, a_0 is arbitrary and $b_0 = c_0 = -1$. This branch exists iff $C < 0$.

(iii) $x = a_0\tau^{-1}$, $y = b_0\tau^p$ and $z = c_0\tau^{-1}$, where $p = -1 - C > -1$ and $a_0 = -C$, b_0 is arbitrary, $c_0 = -1$. Again this branch exists iff $C < 0$.

(iv) $x = a_0\tau^{-1}$, $y = b_0\tau^{-1}$ and $z = c_0\tau^p$, where $p = -2/C > -1$ and $a_0 = -1$, $b_0 = -1/C$, c_0 is arbitrary. This branch exists iff $C < 0$ or $C > 2$.

Step 2. We find the following resonances:

(i) $-1, C/2, (2-C)/2$. (ii)–(iv) $-1, 0, 1$.

Thus, the only cases for which system (11) is weak Painlevé with natural denominator two are $C = \pm 1$ and ± 4 [20].

For $C = 1$ system (11) has only the first singular branch (i) above and step three of the Painlevé analysis yields

$$x = \tau^{-1}(a_0 + a_1\tau^{1/2} + \cdots)$$
$$y = \tau^{-1}(b_0 + b_1\tau^{1/2} + \cdots)$$
$$z = \tau^{-1}(c_0 + c_1\tau^{1/2} + \cdots)$$

where $a_1+b_1+c_1 = 0$. Hence the equations possess algebraic singularities and the full Painlevé property cannot be established. However, if we integrate equations (11) in the complex t-plane for $C = 1$ we find strong evidence of FSS, which suggests that the system may indeed be Painlevé integrable by an appropriate change of coordinates.

Let us now show that (11) is indeed integrable for $C = 1$. Equation (11c) gives

$$x + y = \frac{\dot{z}}{z} \tag{12}$$

where a dot denotes differentiation with respect to t. Differentiating once (11c) and using (11a), (11b) and (12) we obtain

$$2xy = \frac{\ddot{z}}{z} - \left(\frac{\dot{z}}{z}\right)^2 - \dot{z}. \tag{13}$$

Then we differentiate (13) once and using (11a), (11b), (12) and (13) arrive at the equation

$$\frac{1}{z^2}\frac{d^3z}{dt^3} - 4\frac{1}{z^3}\frac{d^2z}{dt^2}\frac{dz}{dt} + 3\frac{1}{z^4}\left(\frac{dz}{dt}\right)^3 - 3\frac{1}{z}\frac{d^2z}{dt^2} + 3\frac{1}{z^2}\left(\frac{dz}{dt}\right)^2 + 2\frac{dz}{dt} = 0 \tag{14}$$

which can be immediately integrated to give

$$\frac{1}{z^2}\frac{d^2z}{dt^2} - \frac{1}{z^3}\left(\frac{dz}{dt}\right)^2 - \frac{3}{z}\frac{dz}{dt} + 2z - K = 0. \tag{15}$$

Note that if we applied the same procedure to (11) for arbitrary C, we would arrive at an equation analogous to (14), which can be integrated once only if $C = 1$.

Combining the first integral of the system, $(x-y)^2 z = D^2 xy$, with (12), (13), to eliminate x, y we obtain

$$\frac{\dot{z}^2}{z} = \left(\frac{D^2}{2} + 2z\right)\left(\frac{\ddot{z}}{z} - \frac{\dot{z}^2}{z^2} - \dot{z}\right)$$

which, when substituted in (15), leads to the equation

$$\frac{dz}{dt} = z\left[\left(\frac{D^2}{2} + 2z\right) + \sqrt{\left(\frac{D^2}{2} + 2z\right)\left(K + \frac{D^2}{2}\right)}\right]. \tag{16}$$

Let us now make the following change of variables in (16):

$$w^2 = \left(\frac{D^2}{2} + 2z\right)\left(K + \frac{D^2}{2}\right)$$

obtaining

$$\frac{dw}{dt} = \frac{G}{2}\left(\frac{w^2}{G} - \frac{D^2}{2}\right)\left(1 + \frac{w}{G}\right) \tag{17}$$

where we have introduced the auxiliary constant

$$G = \left(K + \frac{D^2}{2}\right).$$

We now make the following NTT:

$$w(t) = -G + u(t)$$
$$u(t) = U(T) \qquad \mathrm{d}t = U(T)^{-1}\,\mathrm{d}T$$

and obtain from (17) the simple Riccati equation

$$\frac{\mathrm{d}U}{\mathrm{d}T} = \frac{1}{2G}U^2 - U + \frac{K}{2}$$

which establishes the full Painlevé property of the equations in these variables as also suggested by our numerical results.

8. The geodesic flow on a triaxial ellipsoid

Here, we analyse a system, which is Arnol'd–Liouville integrable and also appears to have only meromorphic solutions. However, even though it passes the usual Painlevé test it cannot be shown to possess the strong Painlevé property in the sense of having a sufficient number of free parameters. Our analysis shows, indeed, that this system is not ACI, but HSS, as it separates into ellipsoidal coordinates.

Let $0 < D_1 < D_2 < D_3$ and define by

$$Q(0) = \left\{\frac{x_1{}^2}{D_1} + \frac{x_2{}^2}{D_2} + \frac{x_3{}^2}{D_3} = 1\right\}$$

a triaxial ellipsoid with semi-axes $\sqrt{D_1}$, $\sqrt{D_2}$ and $\sqrt{D_3}$. The geodesic flow on this ellipsoid can be obtained from the Hamiltonian

$$H = \tfrac{1}{2}(p_1{}^2 + p_2{}^2 + p_3{}^2)$$

where p_i is the conjugate momentum to x_i, with the constraints

$$(x_1, x_2, x_3) \in Q(0) \qquad \sum_{i=1}^{3} \frac{x_i p_i}{D_i} = 0.$$

This system is completely integrable in the sense of possessing a set of involutive first integrals [21]

$$F_i = p_i{}^2 + \sum_{i \neq j} \frac{(x_i p_j - x_j p_i)^2}{D_i - D_j} \qquad i = 1, 2, 3$$

of which only two are independent due to the relation

$$H = \tfrac{1}{2}(F_1 + F_2 + F_3).$$

Let us now fix these constants of motion and pass to ellipsoidal coordinates λ_1, λ_2 which satisfy

$$\frac{x_i{}^2}{D_i} = \frac{\prod_{k=1}^{2}(D_i - \lambda_k)}{\prod_{i \neq j}(D_i - D_j)}$$

with μ_1, μ_2 their associated momenta. Then the following algebraic relation holds:

$$\mu_k{}^2 = \frac{c_1 \lambda_k (\lambda_k - c_2)}{\prod_{j=1}^{3}(\lambda_k - D_j)}$$

with c_1, c_2 depending on the values of the constants of motion. From the above relations, the following genus two hyperelliptic curve is defined

$$\Gamma : \left\{ w^2 = c_1\lambda(\lambda - c_2)\prod_{j=1}^{3}(\lambda - D_j) \right\}$$

and the equations of geodesic motion become

$$\frac{\lambda_1\,\mathrm{d}\lambda_1}{\sqrt{c_1\lambda_1(\lambda_1 - c_2)\prod_{j=1}^{3}(\lambda_1 - D_j)}} + \frac{\lambda_2\,\mathrm{d}\lambda_2}{\sqrt{c_1\lambda_2(\lambda_2 - c_2)\prod_{j=1}^{3}(\lambda_2 - D_j)}} = 0$$

$$\frac{{\lambda_1}^2\,\mathrm{d}\lambda_1}{\sqrt{c_1\lambda_1(\lambda_1 - c_2)\prod_{j=1}^{3}(\lambda_1 - D_j)}} + \frac{{\lambda_2}^2\,\mathrm{d}\lambda_2}{\sqrt{c_1\lambda_2(\lambda_2 - c_2)\prod_{j=1}^{3}(\lambda_2 - D_j)}} = \mathrm{d}t.$$

From these equations we conclude that the generic invariant manifolds of the system are open subsets of two-dimensional strata of a three-dimensional generalized Jacobian associated with Γ, and the geodesic flow is HSS and not ACI.

Now making the NTT

$$-\lambda_1\lambda_2\,\mathrm{d}T = \mathrm{d}t$$

the above system of equations becomes

$$\frac{\mathrm{d}\lambda_1}{\sqrt{c_1\lambda_1(\lambda_1 - c_2)\prod_{j=1}^{3}(\lambda_1 - D_j)}} + \frac{\mathrm{d}\lambda_2}{\sqrt{c_1\lambda_2(\lambda_2 - c_2)\prod_{j=1}^{3}(\lambda_2 - D_j)}} = \mathrm{d}T$$

$$\frac{\lambda_1\,\mathrm{d}\lambda_1}{\sqrt{c_1\lambda_1(\lambda_1 - c_2)\prod_{j=1}^{3}(\lambda_1 - D_j)}} + \frac{\lambda_2\,\mathrm{d}\lambda_2}{\sqrt{c_1\lambda_2(\lambda_2 - c_2)\prod_{j=1}^{3}(\lambda_2 - D_j)}} = 0$$

and the invariant manifolds associated with this new system are open subsets of the two-dimensional Jac(Γ).

Notice that, in the original x_i, p_i variables, this geodesic flow is described by the following system of ODEs:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = D_3x_2(x_1p_2 - x_2p_1) + D_2x_3(x_1p_3 - x_3p_1) + D_2D_3p_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = D_3x_1(x_2p_1 - x_1p_2) + D_1x_3(x_2p_3 - x_3p_2) + D_1D_3p_2$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = D_2x_1(x_3p_1 - x_1p_3) + D_1x_2(x_3p_2 - x_2p_3) + D_1D_2p_3$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = D_3p_2(x_1p_2 - x_2p_1) + D_2p_3(x_1p_3 - x_3p_1)$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = D_3p_1(x_2p_1 - x_1p_2) + D_1p_3(x_2p_3 - x_3p_2)$$

$$\frac{\mathrm{d}p_3}{\mathrm{d}t} = D_2p_1(x_3p_1 - x_1p_3) + D_1p_2(x_3p_2 - x_2p_3).$$

Let us now turn to the results of the Painlevé analysis of these equations.

Step 1. There is only one leading order behaviour of the solutions, namely

$$x_1 = a_{10}\tau^{-1} \qquad p_1 = b_{10}$$

$$x_2 = a_{20}\tau^{-1} \qquad p_2 = b_{20}$$

$$x_3 = a_{30}\tau^{-1} \qquad p_3 = b_{30}$$

where $\tau = t - t_*$ and

$$
\begin{aligned}
&a_{10} : \text{arbitrary} \\
&{a_{20}}^2 = \frac{D_1 - D_3}{D_3 - D_2}{a_{10}}^2 \qquad {a_{30}}^2 = \frac{D_1 - D_2}{D_2 - D_3}{a_{10}}^2 \\
&b_{10} = \frac{D_1}{(D_1 - D_2)(D_1 - D_3)a_{10}} \\
&b_{20} = \frac{D_2}{(D_2 - D_1)(D_2 - D_3)a_{20}} \\
&b_{30} = \frac{D_3}{(D_3 - D_1)(D_3 - D_2)a_{30}}.
\end{aligned}
$$

Step 2. The resonances are -1 (double root), 0, 1 and 2 (double root).

Step 3. Setting

$$
\begin{aligned}
x_1 &= a_{10}\tau^{-1} + a_{11} + a_{12}\tau + \cdots \\
x_2 &= a_{20}\tau^{-1} + a_{21} + a_{22}\tau + \cdots \\
x_3 &= a_{30}\tau^{-1} + a_{31} + a_{32}\tau + \cdots \\
p_1 &= b_{10} + b_{11}\tau + b_{12}\tau^2 + \cdots \\
p_2 &= b_{20} + b_{21}\tau + b_{22}\tau^2 + \cdots \\
p_3 &= b_{30} + b_{31}\tau + b_{32}\tau^2 + \cdots
\end{aligned}
$$

we find

$$
\begin{aligned}
&a_{11} : \text{arbitrary} \\
&a_{21} = \frac{D_2(D_3 - D_1)a_{10}a_{11}}{D_1(D_2 - D_3)a_{20}} \\
&a_{31} = \frac{D_3(D_1 - D_2)a_{10}a_{11}}{D_1(D_2 - D_3)a_{30}} \\
&b_{11} = b_{21} = b_{31} = 0
\end{aligned}
$$

while two of a_{i2} and b_{i2} are also arbitrary. Furthermore, it is not possible to find any additional free parameters by including logarithmic terms in the above expansions.

Let us observe therefore that the above family is not principal, since it only possesses five free constants and hence one cannot conclude that this system has the full Painlevé property. Thus, the fact that the geodesic flow passes the Painlevé test does not contradict the fact that it is HSS instead of ACI. Moreover numerical analysis verifies that this system always admits a periodic pattern of poles and hence only has meromorphic solutions, in agreement with the above results.

It would be interesting to apply to this system the perturbative Painlevé test of Conte *et al* [22] to firmly establish whether this geodesic flow has the strong Painlevé property or not. Such a study is currently under investigation and results will be reported elsewhere.

9. Conclusions

As is well known the term integrability, in the case of dynamical systems, can be given different meanings, depending on the kind of systems studied. In the case of Hamiltonian systems, of course, Arnol'd–Liouville integrability is especially meaningful since it implies for n-degree-of-freedom systems the existence of n independent integrals, including the Hamiltonian, which are in involution with each other and can be used, in principle, to integrate the equations of motion by quadratures.

Given that one is interested in studying actual solutions, the Painlevé property has often proved particularly useful, as it identifies systems whose solutions are all single-valued (not possessing movable critical points) and can often be explicitly integrated in terms of elliptic functions [1, 3]. In fact, a rigorous correspondence has been established between this property and a subclass of integrable systems with rational first integrals through the notion of ACI [6].

Unfortunately not all Arnol'd–Liouville integrable systems with rational integrals are ACI. Thus, an extended class of integrable systems was recently proposed, the so-called HSS, which in general are not ACI, but to which most of the familiar tools of algebraic geometry apply. Many physically interesting examples were already shown to fall into this class, like the Henon–Heiles hierarchy and the Neumann system [7].

In this paper, we have identified a number of additional examples of HSS, in the form of finite-dimensional reductions of a shallow water wave equation and the Harry Dym equation, a Lotka–Volterra system of three competing species and the geodesic flow on a triaxial ellipsoid. The particularly interesting feature of the first three of these examples is that their solutions have only algebraic singularities (hence are locally finitely-sheeted) and are seen to fall into the weak Painlevé class [1, 4]. The geodesic flow passes the usual Painlevé test, but does not have a branch with the full set of free constants, and is not ACI in its original form. Nevertheless, it becomes ACI after a suitable transformation of variables.

Dynamical systems with only algebraic singularities have been conjectured to be completely integrable if their solutions are found to be globally finitely sheeted around arbitrary contours in the complex plane of the independent variable [9, 16, 17]. Such integrations can actually be carried out numerically to great accuracy and with reliable results, using the powerful ATOMFT package developed by Chang and Corliss [15].

Using this numerical procedure, we have shown here that our three weak Painlevé examples indeed appear to possess globally only FSS, exhibiting a periodic lattice of singularities around all the integration paths we have chosen. Based on this evidence and knowing from our geometric-analytical approach that they are HSS, we have been able to find for all of them certain so-called NTTs, first introduced by Goriely [8], mapping them to ACI systems having the strong Painlevé property.

Geometrically, these NTTs act to change one meromorphic differential associated with the system to a holomorphic one, completing the basis of holomorphic differentials associated with a convenient hyperelliptic curve. On the other hand, from the point of view of differential equations, these transformations map the original system to one which is integrable in the sense of Painlevé and may, thus, belong to the class of ACI systems whose properties are well analysed and understood.

Clearly, the interest in finding such transformations is due to the fact that the powerful methods of algebraic geometry can be used to study ACI systems in considerable detail. For instance, all solutions of such systems are meromorphic and, in the Hamiltonian case, complex action-angle variables on complexified Arnol'd–Liouville tori are well defined, discretization through Bäcklund transformations is possible, etc.

The following question, therefore, naturally arises: given a system which possesses only algebraic singularities, under what conditions can we find NTTs that map it to a system that is ACI? All our numerical evidence so far suggests that this may be possible when all solutions along arbitrary contours are globally finitely sheeted, with a periodic pattern of singularities in the complex domain.

However, a complete classification of all classes in which such transformations apply is still lacking. Furthermore, many properties of HSS, like their discretization, still remain to be discovered. We hope to be able to address some of these fascinating issues in a future publication.

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